Differential graded algebras

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October 2024

Contents

1 Introduction

These notes were written for talks in a derived algebraic geometry learning seminar in Fall 2024, following [EP21]. The main goals of these notes will be to explain [EP21, §1-3], mainly discussing differential graded algebras. A large portion of these notes will be copied from my previous notes on differential graded (Lie) algebras, but I'm aiming to expand and sometimes clarify those notes. (Also, I like this template better.)

2 Differential graded algebras

The main idea behind derived algebraic geometry is that derived stuff just works better. Endowing rings of functions with extra structure such as chain complexes gives us more flexibility and structure to work with, and certain families begin to behave better as well. The best formulation is probably using simplicial rings, but it ends up being the same as commutative differential graded algebras in characteristic 0. Another similar theory is spectral algebraic geometry, based on commutative ring spectra.

In this section we'll take a look at differential graded algebras and how they develop the theory of derived algebraic geometry.

We fix a base ring k which is a $\mathbb{Q}$-algebra; therefore, **if k is a field, it has characteristic 0.**

2.1 dg-algebras and affine dg-schemes

The main object of interest is:

Definition 2.1: A **differential graded k-algebra** (also known as dga or dg-algebra) is a chain complex $A = A_\bullet$ of k-modules along with:

- an associative k-linear multiplication $- \cdot - : A_i \times A_j \longrightarrow A_{i+j}$ for all i, j,
- a unit $1 \in A_0$,
- and a k-linear differential $\delta : A_i \longrightarrow A_{i-1}$ for all i, satisfying $\delta^2 = 0$ and $\delta(a \cdot b) = \delta(a) \cdot b + (-1)^{\deg a} a \cdot \delta(b)$.

We can represent a dga A by

$$\cdots \longleftarrow A_{-3} \longleftarrow A_{-2} \longleftarrow A_{-1} \longleftarrow A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow A_3 \longleftarrow \cdots .$$

A graded algebra A is **graded-commutative** if $a \cdot b = (-1)^{(\deg a)\cdot(\deg b)} b \cdot a$ for all homogeneous a, b. A dga which is also graded-commutative is called **differential graded-commutative algebra**, or cdga for short.

We can view a dga $A_\bullet$ in two ways: one way is to view it as a chain complex of k-modules $(A_\bullet, \delta)$ equipped with a multiplication map compatible with the differential δ, and another way is to view it as a graded k-algebra $(A_\bullet, \cdot)$ equipped with a differential δ which is compatible with the multiplication structure.

We generally focus on dga which are **concentrated in nonnegative degree**, i.e. $A_i = 0$ for all $i < 0$, and we'll make that convention here (mainly for the reason that (co)homology in negative degrees are not so important in much of algebraic geometry). As a remark, conventionally algebraic geometers like cochain complexes, but we will use chain complexes here since it is more similar to simplicial objects.

Our first goal is to define the category of k-cdga. We want to mimic the construction of the category of k-algebras, from which we can take the opposite category to define the category of affine k-schemes; this will serve as the blueprint to defining affine k-dg-schemes.

Definition 2.2: Any dga A is a chain complex, so **homology** $H_i(A_\bullet)$ makes sense. If A is a cdga, then $H_\bullet(A_\bullet)$ is itself a cdga, but the differential will be the 0 map (since the elements are by definition killed by the differential).

Remark 2.3: You've likely seen lots of cdgas already, perhaps in the guise of certain cohomology rings. The complexes which compute the (co)homology groups already have a natural multiplication (which is graded-commutative) on them which induces the (graded-commutative) multiplication in the (co)homology rings. Since we are dealing with the (co)homology rings, the differential is just zero and usually dropped from discussion. If, say, k is a field, then any complex is naturally quasi-isomorphic to its (co)homology groups, so the resulting "zero-differential" cdga is quasi-isomorphic to the original one. (Note that depending on if we're dealing with chain complexes of cochain complexes, the cdgas in question will be homologically or cohomologically graded; however this doesn't really change anything we've said.)

Example 2.4: The deRham complex of differential forms (on say, a smooth affine k-scheme Spec R) is a straightforward example. The deRham complex

$$R \longrightarrow \Omega^1_{R/k} \longrightarrow \Omega^2_{R/k} \longrightarrow \cdots \longrightarrow \Omega^{\dim R}_{R/k}$$

has a natural graded-commutative multiplication which is wedge product of two differential forms, and it respects the differential. (Ignore the fact that this is cohomologically graded.) The wedge product induces a multiplication on the cohomology ring $H^\bullet_{dR}(X)$ which is graded-commutative, and the differential is just 0. Hence we have two quasi-isomorphic cdgas: $H^\bullet_{dR}$ with the zero differential is quasi-isomorphic to the deRham complex with the differential given by the one on differential forms and with wedge product as multiplication).

Example 2.5: Another common example is the singular cohomology ring $H^\bullet(X)$. This is computed by the complex of singular cochains, which naturally comes with a multiplication given by the Whitney cup product. Unfortunately, this map **doesn't** respect the differential! Surprisingly, this major issue in topology implies that the complex of singular cochains doesn't form a cdga in the naive way.

Viewing dga's as complexes equipped with a multiplication, we can inherit more notions from being a chain complex.

Definition 2.6: A **morphism of dg-algebras** is a map $f : A_\bullet \to B_\bullet$ which respects the differentials and multiplication. Concretely, $f \circ \delta_A = \delta_B \circ f$ and $f(a \cdot_A b) = f(a) \cdot_B f(b)$. The usual notion of **quasi-isomorphism** (also known as **weak equivalence**) is the same: it is a morphism of dg-algebras which induces isomorphisms on all homology groups. Two cdga $A_\bullet$ and $B_\bullet$ are **quasi-isomorphic** if we have a "roof" diagram (working exactly as in derived categories) $A_\bullet \leftarrow C_\bullet \to B_\bullet$ of quasi-isomorphisms.

Definition 2.7: A dga A is **discrete** if it is concentrated in degree 0, i.e. $A_i = 0$ for all $i \neq 0$.

To any k-algebra A, we associate to it a discrete dga which is just A in degree 0 (and all other terms are 0).

Now let's see an important example of a cdga.

Example 2.8: Let $M_\bullet$ be a graded k-module. The **free graded-commutative k-algebra** generated by $M_\bullet$ is

$$k[M_\bullet] \coloneqq (\mathrm{Sym}^\bullet(M_{\mathrm{even}})) \otimes_k (\Lambda^\bullet(M_{\mathrm{odd}})),$$

where the degree of a tensor of homogeneous elements is just the sum of the degrees of the elements. To make this a cdga, we need to add a differential δ, which we can choose freely on some (homogeneous) basis. (One example is $\delta = 0$.)

Example 2.9: Let $M_\bullet$ be the graded k-module by $M_0 = k\{X\}$ and $M_1 = k\{Y, Z\}$. Then the free graded-commutative algebra $k[M_\bullet]$ is given by

$$k[M_\bullet] = \underbrace{k[X]}_{\deg 0} \oplus \underbrace{k[X] \cdot \{Y, Z\}}_{\deg 1} \oplus \underbrace{k[X] \cdot \{YZ\}}_{\deg 2}.$$

To turn this into a cdga, we need to specify values of $\delta(X), \delta(Y), \delta(Z)$. Since X lives in degree 0, $\delta(X) = 0$. On the other hand Y, Z live in degree 1 so $\delta(Y), \delta(Z) \in k[M_\bullet]_0 = k[X]$, and we have complete freedom to choose such elements. The properties of the differential and multiplication then determine everything else.

In fact, $H_0(k[M_\bullet]) = k[X]/(\delta(Y), \delta(Z))$.

Remark 2.10: It's fairly straightforward to adjust these constructions to differential or analytic geometry. In differential geometry, A_0 should be a C^∞-ring, and the analogue is known as synthetic differential geometry. In analytic geometry, A_0 should be a ring with entire functional calculus (EFC-ring). Pretty much most of the remaining constructions have analogues in such fields, and sometimes are even easier, but we won't reference these similarities (or differences) again. See [EP21] for comments (this part of the notes is based on it anyway).

Now that we've defined the objects and morphisms (and they appear very ordinary), we can go ahead and construct the analogue of the category of k-algebras.

Definition 2.11: We denote by $\mathrm{dg}_+\mathrm{Alg}_k$ the **category of cdga which are concentrated in nonnegative degree**.
The opposite category $(\mathrm{dg}_+\mathrm{Alg}_k)^{\mathrm{op}}$ is the **category of affine dg-schemes**, and is denoted by $\mathrm{DG}_+\mathrm{Aff}_k$. We denote elements in the opposite category by $\operatorname{Spec} A_\bullet$ (this is purely formal at the moment, there is no explicit construction with prime ideals, etc... yet).

Remark 2.12: Although we have not discussed the geometric picture yet, one should visualize the points of $\operatorname{Spec} A_\bullet$ as just the points of $\operatorname{Spec} H_0(A_\bullet)$, which is an ordinary affine scheme. The rest of the structure coming from the higher cohomologies is in some sense infinitesimal. This will be made clearer in §4.3.

2.2 dg-schemes

The next step is to glue affine dg-schemes to form dg-schemes. We might start with the standard: gluing affine dg-schemes along open subschemes, gluing the corresponding modules, etc. But upon further reflection we actually don't really need to, because all of the work is already done for us in standard algebraic geometry – this is exactly what gluing affine schemes and sheaves together is. So we'll just piggyback off of their hard work and define a dg-scheme to just be a chain complex of glued modules over affine schemes – in other words, a chain complex of quasicoherent sheaves.

Definition 2.13: A **dg-scheme** consists of a scheme X^0 along with quasicoherent sheaves $\mathcal{O}_X \coloneqq \{\mathcal{O}_{X,i}\}_{i\geq 0}$ on X^0, satisfying the following conditions:

- $\mathcal{O}_{X,0} = \mathcal{O}_{X^0}$, i.e. the zeroth sheaf is the structure sheaf of the scheme X^0,
- the quasicoherent sheaves are equipped with a cdga structure, consisting of a differential map $\delta : \mathcal{O}_{X,i} \to \mathcal{O}_{X,i-1}$ and a compatible multiplication $- \cdot - : \mathcal{O}_{X,i} \otimes \mathcal{O}_{X,j} \to \mathcal{O}_{X,i+j}$ satisfying the usual conditions.

However, recall that for an affine dg-scheme, the "spectrum" was actually Spec of $H_0(A_\bullet) = \ker(A_1 \xrightarrow{\delta} A_0)$, not all of A_0. So actually the underlying "scheme" is not all of X^0 but the subscheme defined by the ideal $\delta(\mathcal{O}_{X,1}) \subset \mathcal{O}_{X,0}$.

Definition 2.14: Define the **underived truncation** $\pi^0 X \coloneqq \underline{\operatorname{Spec}}_{X^0} H^0(\mathcal{O}_X) \subset X^0$ to be the closed subscheme of X^0 defined by the ideal $\delta(\mathcal{O}_{X,1}) \subset \mathcal{O}_{X,0}$. The underived truncation $\pi^0 X$ is also known as the **classical locus** of X.

Definition 2.15: A morphism of dg-schemes is called a **quasi-isomorphism** if $\pi^0 f : \pi^0 X \to \pi^0 Y$ is an isomorphism of schemes inducing the isomorphisms $\mathcal{H}_\bullet(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{H}_\bullet(\mathcal{O}_X)$ of quasicoherent sheaves on $\pi^0 X = \pi^0 Y$ (these are the homology objects taken in the category of sheaves).

Now the problem comes out with $X^0 \neq \pi^0 X$. The scheme that we actually care about is not all of X^0, so actually in our definition of a dg-scheme, we could have replaced X^0 with an open subset containing $\pi^0 X$; this would give us a quasi-isomorphic dg-scheme. So the "rest" of X^0 is sort of meaningless ambient space which is unwieldly and can get in the way when we try to glue schemes together (since it's just sitting there undefined, can be arbitrarily large, and providing no structure at all). This ends up being too restrictive. The solution is to use derived schemes.

Definition 2.16: A **derived (k-)scheme** X is a scheme $\pi^0 X$ and a presheaf $\mathcal{O}_X$ on the site of affine open subschemes of $\pi^0 X$, taking values in $\mathrm{dg}_+\mathrm{Alg}_k$, such that in degree zero we have $H_0(\mathcal{O}_X) = \mathcal{O}_{\pi^0 X}$, and the $H_i(\mathcal{O}_X)$ are quasicoherent $\mathcal{O}_{\pi^0 X}$-modules for all i.

Let me explain it more concretely. First, we are dropping the ambient scheme X^0 in favor of just the underived truncation $\pi^0 X$, which handles the issue from before. Usually a k-scheme would consist of algebras living over each affine open subscheme, satisfying certain compatibility/gluing. Now we have a dga living over each affine open subscheme, again satisfying certain compatibility/gluing, except that now only the zeroth part $H_0(\mathcal{O}_X)$ actually needs to identify with the structure sheaf of our underlying scheme $\pi^0 X$, and the higher order terms do not. What the higher order terms $H_i(\mathcal{O}_X)$ *do* need to do, is to satisfy quasicoherent compatibility with the structure sheaf $\mathcal{O}_{\pi^0 X} = H_0(\mathcal{O}_X)$:

namely, for each inclusion $U \hookrightarrow V$ of affine open subschemes of $\pi^0 X$ (this is the data of a presheaf on the site of affine open subschemes of $\pi^0 X$), we need quasi-isomorphisms of presheaves of homology groups

$$\mathcal{O}_{\pi^0 X}(U) \otimes^{\mathbf{L}}_{\mathcal{O}_{\pi^0 X}(V)} H_i(\mathcal{O}_X(V)) \xrightarrow{\sim} H_i(\mathcal{O}_X(U)).$$

Note that $\mathcal{O}_X$ refers to the presheaf taking values in $\mathrm{dg}_+\mathrm{Alg}_k$, **not** the structure sheaf of X (as it is commonly used), for an important reason: there is no scheme X here!

Still, dg-schemes and derived schemes are not that far from each other.

Construction 2.17: From a dg-scheme $(X^0, \mathcal{O}_X)$, we get a derived scheme $(\pi^0 X, i^{-1}\mathcal{O}_X)$ where $i : \pi^0 X \hookrightarrow X$ is the canonical embedding. So **any dg-scheme will give us a derived scheme** essentially by "only remembering the classical locus."
On the other hand, from a derived scheme $(\pi^0 X, \mathcal{O}_X)$, we get an affine dg-scheme $\operatorname{Spec} \mathcal{O}_X(U)$ for any affine subscheme $U \subset \pi^0 X$ whose underlying schemes are $\operatorname{Spec}(\mathcal{O}_X(U))_0 \supseteq U$. Unfortunately, these carry the "extraneous space" discussed before, and this lingering ambience actually generally prevents us from being able to glue them together into a globalized scheme $X^0 \supseteq \pi^0 X$.
In simpler terms: imagine a dg-scheme is an apple and the fruit inside (without the skin) is the underived truncation. Then by peeling the skin, we are left with only the juicy goodness of the fruit itself, i.e. a derived scheme. But when presented with peeled apple slices, we may never know what the original skin looked like. So we can't "unscramble the egg" by turning a derived scheme back into a dg-scheme.

Remark 2.18: There are characterizations using sheaves instead of presheaves, and can be obtained from the data in our definition $(\pi^0 X, \mathcal{O}_X)$ by sheafifying each presheaf $\mathcal{O}_{X,n}$ individually. But this messes up some hypersheaf property, so the quasi-inverse functor is not just the forgetful functor of forgetting the sheaf property, which is why we don't use sheaves directly here.

Remark 2.19: According to [EP21], derived schemes as defined here are equivalent to derived Artin or Deligne-Mumford ∞-stacks whose underlying derived stacks are schemes, as in [**?**].

2.3 Quasicoherent complexes

We've discussed dg-algebras, but we've put off dg-modules for a while. The reason is because we actually care about quasicoherent sheaves on derived schemes, rather than dg-schemes. But now that we have derived schemes, let's review dg-modules.

Definition 2.20: Let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$. An **$A_\bullet$-module (in chain complexes)** is a chain complex $M_\bullet$ of k-modules with a corresponding action of $A_\bullet$.
Explicitly, for all i, j we have a k-bilinear map $A_i \times M_j \to M_{i+j}$ satisfying the usual properties of multiplication, and additionally the chain map condition $\delta_M(am) = \delta_a(a)m + (-1)^{\deg a} a\delta_M(m)$.
This is succinctly summarized by giving a map of the total complex $A_\bullet \otimes_k M_\bullet \to M_\bullet$, compatible with the multiplication on $A_\bullet$.
We denote the category of $A_\bullet$-modules concentrated in nonnegative degree by $\mathrm{dg}_+\mathrm{Mod}_{A_\bullet}$. We have the usual definition of morphisms of $A_\bullet$-modules and quasi-isomorphism.

Now we globalize this.

Definition 2.21: Let $(\pi^0 X, \mathcal{O}_X)$ be a derived scheme. The analogue of the previous definition is to look at $\mathcal{O}_X$-modules $\mathcal{F}$ in complexes of presheaves. We say they are **quasicoherent complexes** (sometimes called **homotopy Cartesian modules**) if the homology presheaves $H_i(\mathcal{F})$ are all quasicoherent $\mathcal{O}_{\pi^0 X}$-modules.

Once again, the condition that all $H_i(\mathcal{F})$ are quasicoherent $\mathcal{O}_{\pi^0 X}$-modules is just saying that for every inclusion $U \hookrightarrow V$ of affine open subschemes of $\pi^0 X$, we have quasi-isomorphisms

$$\mathcal{O}_X(U) \otimes^{\mathbf{L}}_{\mathcal{O}_X(V)} \mathcal{F}(V) \xrightarrow{\sim} \mathcal{F}(U).$$

2.4 Missing pieces

The intuition from derived categories tells us that we should think of derived schemes X, Y as equivalent if they can be connected by a zigzag of roofs of quasi-isomorphisms:

$$\begin{tikzcd} & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ X & & \bullet & & \bullet & & \bullet & & Y \end{tikzcd}.$$

But that raises the question: how should we define morphisms to be compatible with this notion of equivalence? How should we define gluing? There are many issues that arise, including a glaring one which is that forcibly inverting quasi-isomorphisms gives a "homotopy category" which doesn't have limits and colimits and doesn't behave well with gluing, even in the affine case. If we take I to be the poset of open affine subschemes (as in the definition of a derived scheme), then we could try to form $\mathrm{dg}_+\mathrm{Alg}_k^I$ the category of I-shaped diagrams in $\mathrm{dg}_+\mathrm{Alg}_k$. Then we can invert quasi-isomorphisms to obtain the homotopy category $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k^I)$ of $\mathrm{dg}_+\mathrm{Alg}_k^I$, but unfortunately the natural functor $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k^I) \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)^I$ is not an equivalence (in fact it fails for everything but discrete diagrams, i.e. when I is just a set). The tl;dr is that **lots of problems arise when trying to do the standard constructions (i.e. homotopy category, derived category) to define the right category for derived schemes**. Actually the answer is to use ∞-categories, specifically $(\infty, 1)$-categories.

3 Infinity categories, model categories, and consequences for dg-algebras

I don't really want to get into the formality of infinity categories and model categories, so I'll try to minimize the abstractness as much as possible and only bring up the necessary conventions so we can look at some consequences for dg-algebras.

3.1 Pretending to understand infinity categories

There are many equivalent notions of ∞-categories, and as such, there are many ways to get an extremely superficial understanding of them. Let's see a few for intuition.

(1) Perhaps the easiest notion is the concept of a **topological category**, which is a category enriched in topological spaces. Concretely, this just means that we should equip the Hom-sets of a category with the structure of a topological space, so that we have additional information (namely topological information) when dealing with morphisms. Naturally everything should now be phrased in terms of continuity.

    The **homotopy category** $\mathrm{Ho}(C)$ of a topological category C is just the category where we only remember the path components of morphisms, i.e. $\mathrm{Obj}(\mathrm{Ho}(C)) = \mathrm{Obj}(C)$, but $\mathrm{Hom}_{\mathrm{Ho}(C)}(X, Y) = \pi_0 \hom_C(X, Y)$.

    A functor $\mathcal{F} : C \to \mathcal{D}$ of topological categories (of course, assumed to respect topological structure) is a **quasi-equivalence** if F induces isomorphisms on all homotopy groups $\pi_n(\mathrm{Hom}_C(X, Y)) \xrightarrow{\sim} \pi_n(\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ for all X, Y, as well as an equivalence of homotopy categories (this basically means it's essentially surjective and also induces bijections on π_0 of the Hom-spaces). The upshot is that for functors to be "equivalences" they only need to be weak equivalences on Hom-sets (or rather Hom-spaces), and not true homeomorphisms.

(2) Topological spaces carry lots of information which we don't necessarily always want. A much more effort presentation is given by **simplicial categories**, which have a simplicial set of morphisms between objects. They're very similar to topological categories due to the relationship between simplicial sets and topological spaces, but they are much easier to handle.

(3) The easiest notion to construct is called a **relative category**. These are just pairs $(C, \mathcal{W})$ where $\mathcal{W} \subset C$ is a subcategory encoding "some notion of equivalence" which is not as strong as isomorphism. Two very common notions of $\mathcal{W}$ are given by weak equivalences of topological spaces, and quasi-isomorphisms of chain complexes.

The homotopy category $\mathrm{Ho}(C)$ is just the localization of C at $\mathcal{W}$ (note that this is different from the classical notion of homotopy category in homological algebra, in which only "strong homotopy equivalences" are inverted). The simplicial category is closely related. The drawback of using relative categories is that quasi-equivalences are hard to describe.

One point that [EP21] makes is that when given an infinity category, you can think of it as a topological or simplicial category (since they're easier to visualize), while if you need to construct one, you can just give a relative category (since they're easier to construct formally but harder to understand concretely).

3.2 Model categories

Model categories are pretty much just relative categories with extra structure, and this extra structure helps us compute stuff - namely derived functors. The first and foremost point of relative categories is to specify a class of "weak equivalences" which will be more general than isomorphisms, but the correct notion for our situation. The rest of the structure exists to make the category more concrete and accessible, i.e., to do computations.

Definition 3.1: A **model category** is a relative category $(C, \mathcal{W})$ together with two classes of morphisms called **fibrations** and **cofibrations**, which satisfy certain axioms (inspired by algebraic topology).

A fibration which is also lies in $\mathcal{W}$ (i.e., a weak equivalence) is called a **trivial fibration**, and **trivial cofibrations** are defined in the same way.

We'll skip the exact definitions of fibrations and cofibrations, as well as their properties, and get straight to examples. However, we'll point out that we'll use the following notation:

- Weak equivalences will be denoted by $\xrightarrow{\sim}$.
- Fibrations will be denoted by surjections $\twoheadrightarrow$.
- Cofibrations will be denoted by injections $\hookrightarrow$.

Example 3.2 (trivial model structure): Suppose C is a category with limits and colimits. Then the **trivial model structure** is just the model structure where weak equivalences are isomorphisms, and fibrations and cofibrations are just all morphisms.

Example 3.3 ($\mathrm{dg}_+\mathrm{Alg}_k$): One important model structure on $\mathrm{dg}_+\mathrm{Alg}_k$ is as follows.

- Weak equivalences are quasi-isomorphisms.
- Fibrations are maps which are surjective in strictly positive degree, i.e. maps $A_\bullet \to B_\bullet$ such that $A_i \twoheadrightarrow B_i$ for all $i > 0$.
- Cofibrations are maps $A_\bullet \to B_\bullet$ which have the left lifting property with respect to trivial fibrations, i.e. for any trivial fibration $X_\bullet \to Y_\bullet$ then for any commutative square

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, "f"'] & X_\bullet \arrow[d] \\ B_\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & Y_\bullet \end{tikzcd}$$

  there should exist a map $B_\bullet \to X_\bullet$ still making the diagram commute.
  A more concrete way to understand cofibrations are as retracts of quasi-free maps. A quasi-free map is a map $A_\bullet \to B_\bullet$ such that $B_\bullet$ is freely generated as a graded-commutative $A_\bullet$-algebra.

Example 3.4 ($\mathrm{DG}_+\mathrm{Aff}_k$): Since $\mathrm{DG}_+\mathrm{Aff}_k = (\mathrm{dg}_+\mathrm{Alg}_k)^{\mathrm{op}}$, we give it the opposite model structure, which swaps fibrations and cofibrations.

Example 3.5 (Complexes of R-modules in nonnegative degree): Consider the category of nonnegatively graded chain complexes of R-modules. The **projective model structure** is given as follows.

- The weak equivalences are quasi-isomorphisms.
- The fibrations are chain maps which are surjective in strictly positive degree.
- The cofibrations are maps $M_\bullet \hookrightarrow N_\bullet$ such that $N_\bullet / M_\bullet$ is a complex of projective R-modules.

The homotopy category is the full subcategory of $D(R\text{–mod})$ of nonnegatively graded chain complexes.

Example 3.6 (Complexes of R-modules in nonpositive degree): There is also an **injective model structure** similar to the previous example.

- The weak equivalences are quasi-isomorphisms.
- The cofibrations are chain maps which are injective in strictly negative degree.
- The fibrations are surjective maps whose levelwise kernels are all injective modules.

The homotopy category is the full subcategory of $D(R\text{–mod})$ of nonpositively graded chain complexes.

Example 3.7 (All complexes of R-modules): The classical picture is to **construct the derived category $D(R)$ by inverting all quasi-isomorphisms amongst all chain complexes**. The previous two examples handle the cases where the chain complexes are concentrated entirely in nonnegative or nonpositive degree. To generalize to all complexes, we just need to put more restrictions on either the cofibrations or fibrations, but the end result is that the homotopy category is indeed $D(R\text{–mod})$.

The upshot of defining these model structures is that we're able to compute stuff using the notions of fibrations and cofibrations.

Definition 3.8: Let C be a model category. An object $X \in C$ is **fibrant** if the map to the final object $X \to f$ is a fibration; it is **cofibrant** if the map from the initial object $i \to X$ is a cofibration.
Oftentimes we want to work with fibrant and cofibrant objects rather than arbitrary objects. Therefore we need to replace some object $X \in C$ with an equivalent (co)fibrant object. A **fibrant replacement** of A is some fibrant object $\widehat{A}$ with a weak equivalence $A \to \widehat{A}$. A **cofibrant replacement** of A is some cofibrant object $\widetilde{A}$ with a weak equivalence $\widetilde{A} \to A$.

Example 3.9 (I): Example 3.3 we gave a model structure on $\mathrm{dg}_+\mathrm{Alg}_k$. With this model structure, every object is fibrant.

The **main intuition for (co)fibrant replacements is injective and projective resolutions.**

Example 3.10: In the projective model structure (on nonnegatively graded chain complexes of R-modules) from Example 3.5, the 0 complex is both the initial and terminal object. Now **everything is fibrant**, since every complex surjects (level-wise) onto the zero complex. On the other hand, **cofibrant objects are precisely the complexes of projective modules**. Therefore a cofibrant replacement is precisely a projective resolution!

Example 3.11: The same thing happens for the injective model structure in Example 3.6. Everything is cofibrant, while the fibrations are complexes of injective modules. Therefore **fibrant replacement is just an injective resolution**.

Definition 3.12 (path object): Given a fibrant object X, a **path object** PX for X is an object PX, together with a diagram

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[dr, "\mathrm{diag}"'] & PX \arrow[d, two heads, "f"] \\ & X \times X \end{tikzcd}$$

where the top map is a weak equivalence and f is a fibration.

Remark 3.13: Path objects always exist, as a consequence of a certain axiom of model categories.

The reason path objects are useful is the following:

Theorem 3.14 (Quillen): Let $A \in \mathcal{C}$ be a cofibrant object and X a fibrant object. Then the space of morphisms $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(A, X)$ in the homotopy category $\mathrm{Ho}(\mathcal{C})$ are given by the co-equalizer of the diagram

$$\mathrm{Hom}_{\mathcal{C}}(A, PX) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(A, X),$$

where the two maps are induced by the two projections

$$PX \to X \times X \rightrightarrows X.$$

3.3 Derived functors in model categories

One important use of model structures is to give conditions for the existence of **derived functors** and methods for computing them.

Definition 3.15: A functor $G : \mathcal{C} \to \mathcal{D}$ of model categories is **right Quillen** if it has a left adjoint F (i.e., it is a right adjoint) and preserves fibrations and trivial fibrations.
Duually, F is **left Quillen** if it has a right adjoint (i.e., it is a left adjoint) and F preserves cofibrations and trivial cofibrations.
In this case, $F \dashv G$ is a **Quillen adjunction**.

The intuition for these comes from **right exact and left exact functors**. In the case of the derived category, left adjoints are right exact and give rise to left derived functors. **In the case of model categories, left Quillen functors are left adjoints and give rise to left-derived functors** and similarly for right Quillen.

Lemma 3.16: Let $F \dashv G$ be an adjunction of functors of model categories. F is left Quillen iff G is right Quillen.

Theorem 3.17: If G is right Quillen, then the **right-derived functor** $\mathbf{R}G$ exists and sends $A \mapsto G(\widehat{A})$ where $\widehat{A}$ is a fibrant replacement.
If F is left Quillen, then the **left-derived functor** $\mathbf{L}G$ exists and sends $A \mapsto F(\widetilde{A})$ where $\widetilde{A} \to A$ is a cofibrant replacement.

Remark 3.18: If you don't like the arbitrary choice of a fibrant replacement, we can even take fibrant replacements functorially, but since right Quillen functors preserve weak equivalences between fibrant objects, it turns out this isn't strictly necessary (on objects at least).

Example 3.19: The global sections functor Γ is left-exact and is a right-adjoint to Spec. The standard way to define $\mathbf{R}\Gamma$ (which computes sheaf cohomology) is to take an injective resolution. But actually fibrant replacement in the model category of nonnegatively graded cochain complexes is the same thing as taking an injective resolution - so it turns out that the classical procedure exactly matches our procedure of taking a fibrant replacement to the right Quillen functor Γ!

4 Some consequences for dg-algebras

Now that we are armed with quite a lot of abstract nonsense, let's see what it can do when applied to dg-algebras. We want to use the model structure from Example 3.3.

The first step is noting the embedding $\mathrm{Alg}_k \hookrightarrow \mathrm{dg}_+\mathrm{Alg}_k$ by embedding each algebra as a chain complex concentrated in degree 0. We also have a map $\mathrm{dg}_+\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ by passing to the homotopy category.

Lemma 4.1: The composition $\mathrm{Alg}_k \hookrightarrow \mathrm{dg}_+\mathrm{Alg}_k \to \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ is fully faithful.

In other words, for an (ordinary) affine scheme X and a derived affine scheme Y, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{DG}_+\mathrm{Aff}_k)}(X, Y) \cong \mathrm{Hom}_{\mathrm{Aff}_k}(X, \pi^0 Y).$$

(A similar statement holds for non-affine X, Y.)

Proof. First, let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ and $B \in \mathrm{Alg}_k$. Then note that for any $f \in \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(A_\bullet, B)$, that $f : A_{>0} \to B_{>0} = 0$. In particular, since f commutes with the differential δ, we have

$$f(\delta(a)) = \delta(f(a)) = 0 \quad \forall\, a \in A_{>0}.$$

Thus

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(A_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(A_\bullet), B).$$

In particular we can replace $A_\bullet$ with anything weakly equivalent (i.e. quasi-isomorphic) to it, especially say a cofibrant replacement $\widetilde{A}_\bullet$, to get

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}_\bullet, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(A_\bullet), B).$$

Now we want to compute

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)}(A, B)$$

for $A, B \in \mathrm{Alg}_k$, and show that it is equal to $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B)$. First we can take a cofibrant replace $\widetilde{A}_\bullet \to A$, and weak equivalence implies that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)}(A, B) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)}(\widetilde{A}_\bullet, B).$$

Now $\widetilde{A}_\bullet$ is cofibrant and in the model structure everything in fibrant (in particular B is fibrant), so Theorem 3.14 implies that

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)}(\widetilde{A}_\bullet, B) = \mathbf{coeq}\left(\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}_\bullet, PB) \rightrightarrows \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}, B)\right).$$

Claim 4.2: One choice for PB is B.

Proof. We need an object PB which is weakly equivalent, i.e. quasi-isomorphic, to B, and the map $PB \twoheadrightarrow B \times B$ is a fibration. But $B \times B$ is a complex concentrated in degree 0, so there's nothing in positive degrees; in particular **every** morphism of complexes to $B \times B$ is a fibration, so we're free to choose anything! Therefore we just need something quasi-isomorphic to B, and a map to $B \times B$ whose composition with this quasi-isomorphism is the diagonal map $B \to B \times B$. The easiest choice is just $PB = B$ and the map $PB \to B \times B$ is once again the diagonal map. □

Now the composition of the two maps

$$B \xrightarrow{\text{diag}} B \times B \rightrightarrows B$$

are both identity, therefore the coequalizer of the diagram

$$\mathbf{coeq}\left(\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}_\bullet, PB) \rightrightarrows \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}, B)\right) = \mathbf{coeq}\left(\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}_\bullet, B) \rightrightarrows \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}, B)\right) = \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}, B).$$

But we computed this already: we have

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(\widetilde{A}, B) = \mathrm{Hom}_{\mathrm{Alg}_k}(H_0(\widetilde{A}_\bullet), B) = \mathrm{Hom}_{\mathrm{Alg}_k}(A, B).$$

□

Another very important consequence of the model structure on $\mathrm{dg}_+\mathrm{Alg}_k$ is how derived functors work. In the next subsection we'll see how derived tensor products work.

4.1 Derived tensor products

If $A_\bullet, B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ then we can treat them simply as complexes, forgetting the algebra structure on them. There is already the notion of the tensor product of complexes, which is the complex $(A_\bullet \otimes_k B_\bullet)_\bullet$ whose nth graded component is $\sum_{i+j=n} A_i \otimes_k B_j$, and it is not difficult to specify the differential. It turns out there is no problem defining a multiplication structure either: just $(a \otimes b) \cdot (a' \otimes b') = (-1)^{(\deg a')(\deg b)}(aa' \otimes bb')$. Therefore we have a notion of tensor product of dgas, which agrees with the tensor product of complexes when we forget the algebra structure.

Our goal is to define the **derived tensor product**, in much the same way as we define it in the derived category (which is by taking projective resolutions of both factors and then applying the usual tensor product of complexes to these resolutions).

The first step in defining a derived functor is to check whether it is right Quillen or left Quillen.

Lemma 4.3: The functor $-\otimes_k- : \mathrm{dg}_+\mathrm{Alg}_k \times \mathrm{dg}_+\mathrm{Alg}_k \longrightarrow \mathrm{dg}_+\mathrm{Alg}_k$ is left Quillen. It has a right adjoint $A \mapsto (A, A)$.

Proof. It's obvious $A \mapsto (A, A)$ is the correct candidate for a right adjoint: by the universal property of tensor product,

$$\mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}((A \otimes_k B), C) \simeq \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(A, C) \times \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k}(B, C) \simeq \mathrm{Hom}_{\mathrm{dg}_+\mathrm{Alg}_k \times \mathrm{dg}_+\mathrm{Alg}_k}((A, B), (C, C)).$$

Now we can check that this functor is right Quillen. It clearly preserves fibrations, as $C \twoheadrightarrow D$ (i.e. $C_i \twoheadrightarrow D_i$ for all $i > 0$) iff $C \times C \twoheadrightarrow D \times D$. It also clearly preserves quasi-isomorphisms, hence it preserves trivial fibrations. Therefore $A \mapsto (A, A)$ is right Quillen. By Lemma 3.16, this automatically implies that $- \otimes_k -$ is left Quillen. □

Now the general theory of §3.3 (specifically, Theorem 3.3) implies that the left derived functor $- \otimes_k^{\mathbf{L}} -$ exists, and gives an explicit description of it.

Definition 4.4: The **derived graded tensor product** $\otimes_k^{\mathbf{L}} : \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k) \times \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k) \longrightarrow \mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ is defined to be the left derived functor of $\otimes_k$.
Explicitly, to compute $A_\bullet \otimes_k^L B_\bullet$, take cofibrant replacements $\widetilde{A}_\bullet$ and $\widetilde{B}_\bullet$ and then take the usual tensor product:

$$A_\bullet \otimes_k^L B_\bullet = \widetilde{A}_\bullet \otimes_k \widetilde{B}_\bullet.$$

Remark 4.5: Recall that k here can be any $\mathbb{Q}$-algebra, not just a field, so this construction is not as obvious as it might seem.

Actually, when we take derived tensor products in a derived category of coherent sheaves, we don't take projective resolutions of the factors (in large part because they usually don't exist): we get away with the much easier procedure of taking a locally free resolution. The same general principle holds here: **we don't actually need cofibrant replacements**, but rather something much less complicated.

Definition 4.6: Let $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ be a cdga and $M_\bullet$ an $A_\bullet$-module. Then $M_\bullet$ is called **quasi-flat** if the underlying graded module of $M_\bullet$ is flat over the the underlying graded algebra of $A_\bullet$.

In other words we want to just forget the differential and check if we have flatness as graded modules.

Proposition 4.7: To compute $A_\bullet \otimes^{\mathbf{L}}_k B_\bullet$, we can take a quasi-flat replacement (over k) of either of the two factors, then apply the usual tensor product $- \otimes_k -$.

In particular, if $A_\bullet$ is a complex of flat k-modules, then $(A \otimes^{\mathbf{L}}_k B)_\bullet$ can be computed by the ordinary graded tensor product $(A \otimes_k B)_\bullet$.

Proof. The essential part is that if $\widetilde{A}_\bullet, \widetilde{B}_\bullet$ are cofibrant replacements of $A_\bullet, B_\bullet$, then $\widetilde{A}_\bullet, \widetilde{B}_\bullet$ are quasi-flat over k (due to being a retract of a quasi-free k-module). We just need to know that the homology groups of

$$(A \otimes^{\mathbf{L}}_k B)_\bullet \coloneqq (\widetilde{A} \otimes_k \widetilde{B})_\bullet$$

match the homology groups of

$$(M \otimes_k B)_\bullet \quad \text{or} \quad (A \otimes_k N)_\bullet,$$

where M is a quasi-flat replacement of A and N is a quasi-flat replacement of B. But being quasi-flat means that the tensor product is exact, and being a replacement means they're quasi-isomorphisms; in particular they're quasi-flat replacements of $\widetilde{A}_\bullet$ and $\widetilde{B}_\bullet$. Hence the homology groups of the replacements match the homology groups of $(\widetilde{A} \otimes_k \widetilde{B})_\bullet$, but that's exactly $(A \otimes^{\mathbf{L}}_k B)_\bullet$. □

This holds more generally over an arbitrary base $R_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$, not just an honest ring k:

Proposition 4.8: To compute $A_\bullet \otimes^{L}_{R_\bullet} B_\bullet$ we can take a quasi-flat replacement of either of the two factors, then apply the usual tensor product $- \otimes_{R_\bullet} -$.

Recall that the tensor product in the category of k-algebras plays the dual role to the fiber product in the category of k-schemes. This is still true in $\mathrm{dg}_+\mathrm{Alg}_k$:

Definition 4.9: In the opposite category $\mathrm{DG}_+\mathrm{Aff}_k$, we denote these derived tensor products as **homotopy pullbacks**, and we write $X \times^h_Z Y \coloneqq \operatorname{Spec}(A_\bullet \otimes^{L}_{C_\bullet} B_\bullet)$ where $X = \operatorname{Spec} A_\bullet$, $Y = \operatorname{Spec} B_\bullet$, and $Z = \operatorname{Spec} C_\bullet$.

Example 4.10: Let's compute the homotopy pullback

$$\{0\} \times^h_{\mathbb{A}^1} \{0\},$$

which is just Spec of

$$k \otimes^{\mathbf{L}}_{k[t]} k.$$

In the classical setting, this is the fiber product of the two maps $\{0\} \hookrightarrow \mathbb{A}^1 \hookleftarrow \{0\}$, which is clearly just $\{0\} \cong \operatorname{Spec} k$ (alternatively, this is easily computed as $\operatorname{Spec} k \otimes_{k[t]} k \cong \operatorname{Spec} k$). But in the derived setting, this is quite interesting!

First we need a quasi-flat resolution of k over $k[t]$. This is given by the cdga $k[t,s]$ where $\deg t = 0$ and $\deg s = 1$ and $\delta(s) = 1$, so that the complex is $k[t] \cdot s \to k[t]$, and $s \mapsto t$.

Remark 4.11: In fact, this is even a cofibrant replacement. The cdga $k[t,s]$ with $\delta(s) = t$, $\deg t = 0$, and $\deg s = 1$, is freely generated over $k[t]$, i.e. quasi-free, hence cofibrant. On the other hand $k[t,s] \to k$ is a quasi-isomorphism. Therefore $k[t,s] \to k$ is a cofibrant replacement.

Now we can apply the usual tensor product of dgas:

$$k \otimes^{L}_{k[t]} k = k[t,s] \otimes_{k[t]} k = k[s] = (k\{s\} \xrightarrow{s \mapsto t} k).$$

But in degree 0, $t = 0$, so the complex is

$$k \xrightarrow{0} k.$$

Note that applying the homology functor H_0 to the complex recovers k, which indeed agrees with the underived fiber product (i.e. tensor product of rings). What the **derived** tensor product is telling us here is that there are "virtual points" in the derived scheme $\operatorname{Spec} k[s]$ which may have positive or negative weights depending on which graded component they show up in. Applying the Euler characteristic to the complex $k \xrightarrow{0} k$, we obtain $1 - 1 = 0$, so the derived intersection "should" contain "zero total points." But that means that there is the negatively weighted "virtual point" counteracting our "legitimate" point given in degree 0 that we can recover by applying H_0!

We can also talk about the "virtual dimension," given informally by the Euler characteristic of the generators. Here, $k[s]$ has a single generator in degree 1 (which is odd), so the virtual dimension is -1. This agree with the notion of "intersecting two codimension 1 subschemes of a dimension 1 scheme," which realistically "should" be dimension $1 - 1 - 1 = -1$. Of course, in the underived setting it's dimension 0 (and negative dimension doesn't make sense anyway).

Remark 4.12: These properties are instances of a more general phenomenon of generalizing the properties from the classical world to the derived world.

Example 4.13: More generally, let's consider the derived intersection

$$\{a\} \times^h_{\mathbb{A}^1} \{0\} = \operatorname{Spec}(k \otimes_{a \leftarrow t, k[t], t \mapsto 0} k).$$

In the underived setting, this is just $\operatorname{Spec} 0 = \emptyset$ whenever $a \neq 0$, and $\operatorname{Spec} k = \{0\}$ when $a = 0$. It's kind of weird that this has a different size as a varies. Indeed, this is exactly a strange issue that comes up in intersection theory: when we multiply two divisors, we need to swap out one of the divisors for a linearly equivalent divisor which "intersects transversely" in order to count it geometrically (which always seemed pretty unsatisfactory to me).

Now let's consider the derived setting. We have the quasi-flat (even cofibrant) $k[t]$-algebra $k[s, t]$ where $\deg t = 0$, $\deg s = 1$, and $\delta(s) = t - a$. This is quasi-isomorphic to k as a $k[t]$-module corresponding to $k = k[t]/(t - a)$. Now we compute the derived intersection as

$$k \otimes_{a \leftarrow t, k[t], t \mapsto 0} k = k[t, s] \otimes_{a \leftarrow t, k[t], t \mapsto 0} k[t]/(t) = k[s],$$

where $\delta(s) = t - a$. But here $t = 0$, so $\delta(s) = -a$, and our derived intersection complex is

$$k \xrightarrow{1 \mapsto -a} k.$$

Now, when $a = 0$, then we recover the example from above: we get

$$k \xrightarrow{0} k.$$

However, when $a \in k^\times$ is invertible, then $k[s]$ is quasi-isomorphic to the 0 complex, as the map $1 \mapsto a$ is surjective. Hence the derived intersection is quasi-isomorphic to $\operatorname{Spec} 0 = \emptyset$. This obviously has no points. But as we discussed in the example with $a = 0$, the "total" number of points in the derived intersection is still 0! This is because the Euler characteristic of $k[s]$ is always 0, regardless of what $\delta(s)$ is. In particular, this corresponds to our intuition of intersecting two generic points in $\mathbb{A}^1$, which should generically be empty. Consequently, it categorifies Serre's intersection numbers, giving a more fulfilling answer to why intersection numbers behave the way they do.

Our last example is the important example of a **derived loop space**.

Definition 4.14: Let $X \in \mathrm{DG}_+\mathrm{Aff}_k$. The **derived loop space** of X is

$$\mathcal{L}X \coloneqq X \times^h_{X \times X} X,$$

where the two maps are the diagonal embeddings $\Delta : X \hookrightarrow X \times X$.

Remark 4.15: These loop spaces don't look like loop spaces in topology, because the notion of equivalence is totally different.

Example 4.16: Let's compute $\mathcal{L}\mathbb{A}^1 \coloneqq \mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1$. This is the Spec of

$$(k[x,y]/(x-y) \otimes^{\mathbf{L}}_{k[x,y]} (k[x,y]/(x-y)).$$

We need a quasi-flat replacement of $k[x,y]/(x-y)$ as a $k[x,y]$-module. One example we can take is $k[x,y,s]$ with $\deg x = \deg y = 0$, and $\deg s = 1$, with $\delta(s) = x - y$ (this is even cofibrant, check this!). This gives us the complex

$$k[x,y] \cdot s \longrightarrow k[x,y].$$

Now we apply the usual tensor product to find that

$$\mathcal{L}\mathbb{A}^1 = \mathbb{A}^1 \times^h_{\mathbb{A}^1 \times \mathbb{A}^1} \mathbb{A}^1 = \mathrm{Spec}(k[x,y]/(x-y) \otimes^{\mathbf{L}}_{k[x,y]} k[x,y,s]) = \mathrm{Spec}\, k[x,s],$$

where $k[x,s]$ is the cdga with $\deg x = 0$, $\deg s = 1$, and $\delta(s) = x - x = 0$. Therefore this is the cdga

$$k[x] \cdot s \xrightarrow{0} k[x],$$

with underlying scheme $\mathrm{Spec}\, H_0(k[x,s]) = \mathrm{Spec}\, k[x]$, but "virtual dimension" $1 - 1 = 0$. This is indeed "expected," given that we're "intersecting" two codimension-1 things in something of dimension 2, hence we "expect" their "intersection" to have dimension $2 - 1 - 1 = 0$. (More generally we expect the virtual dimension of $\mathcal{L}X$ to be zero, as we're intersecting X with X inside $X \times X$; each has codimension equal to $\dim X$, hence the expected dimension is $\dim X \times X - \dim X - \dim X = 0$.)

Example 4.17: Let X be an arbitrary smooth affine scheme $\mathrm{Spec}\, R$. Let us describe $\mathcal{L}X$. This is Spec of

$$R \otimes^{\mathbf{L}}_{R \otimes_k R} R.$$

The homology groups of this are exactly $\mathrm{Tor}^{R \otimes_k R}_\bullet(R,R)$, i.e. the Hochschild homology groups of R! If we wanted to compute this directly, we can use the bar complex to resolve R as an $R \otimes_k R$-module, and then apply $- \otimes_{R \otimes_k R} R$; however, the **HKR isomorphism** tells us that this object is quasi-isomorphic to the complex

$$R \otimes^{\mathbf{L}}_{R \otimes_k R} R \xrightarrow{\sim} \left(R \xleftarrow{0} \Omega^1_{R/k} \xleftarrow{0} \Omega^2_{R/k} \xleftarrow{0} \cdots \xleftarrow{0} \Omega^{\dim R}_{R/k} \right).$$

Therefore

$$\mathcal{L}(\mathrm{Spec}\, R) = \mathrm{Spec} \left(R \xleftarrow{0} \Omega^1_{R/k} \xleftarrow{0} \Omega^2_{R/k} \xleftarrow{0} \cdots \xleftarrow{0} \Omega^{\dim R}_{R/k} \right).$$

4.2 Obstruction theory

Obstruction theory is one area which benefits immensely from derived techniques. Let's first recall how classical obstruction theory works.

Definition 4.18 (dual numbers): For a commutative ring R, we define the **dual numbers** $R[\varepsilon]$ where $\deg(\varepsilon) = 0$ and $\varepsilon^2 = 0$, so that as R-modules, $R[\varepsilon] \simeq R \oplus R\varepsilon$.

Construction 4.19 (tangent vectors): Let X be a smooth k-scheme. The k-points of X, denoted by $X(k)$, are given by maps $\operatorname{Spec} k \to X$. The **tangent vectors** of X are given by maps

$$\operatorname{Spec} k[\varepsilon] \to X,$$

so that

$$X(k[\varepsilon]) \simeq \{(x, v) \mid x \in X(k), \quad v \text{ a tangent vector at } x\}.$$

We think of $\operatorname{Spec} k[\varepsilon]$ as a point with a choice of infinitesimal direction. Therefore a map $\operatorname{Spec} k[\varepsilon] \to X$ is basically a choice of a (k-)point in X, along with a choice of an infinitesimal direction at that point, i.e. a tangent vector v at x.

More generally, for any ring A and any A-module I, we construct the ring $A \oplus I$ by setting $I \cdot I = 0$. Then $X(A \oplus I)$ consists of I-valued tangent vectors at A-valued points of X.

Definition 4.20 (square-zero extension): A square-zero extension of commutative rings is a surjective map $f : A \twoheadrightarrow B$ such that $(\ker f)^2 = 0$ as an ideal in A.

More generally in deformation theory, we want to consider **nilpotent extensions**, which are surjections of rings $A \twoheadrightarrow B$ such that the kernel is a nilpotent ideal. But it's a fact that any nilpotent extension is actually just a composite of finitely many square-zero extensions, which allows us to only consider square-zero extensions.

Here are two motivating examples of obstruction spaces.

Example 4.21: Let X be a smooth k-scheme. Fix a square-zero extension $f : A \twoheadrightarrow B$ with $I \coloneqq \ker f$. Then

$$X(A) \times_{X(B)} X(A) \simeq X(A) \times_{X(B)} X(B \oplus I),$$

because $A \times_B A \simeq A \times_B (B \oplus I)$. In particular I-valued tangent vectors at B-valued points of X act transitively on the fibers of $X(A) \to X(B)$.

Note that $X(A) \to X(B)$ is only surjective if X is smooth (assume X is also finite type). Singularities in X give obstructions to lifting $X(B)$ to $X(A)$. It's long been observed that there are "obstruction spaces" which control the failure of attempting to lift elements in $X(B)$ to $X(A)$; they tend to be some bundle over $X(B)$.

Example 4.22: Here's another example from homological algebra. Let $f : A^\bullet \twoheadrightarrow B^\bullet$ bs a levelwise surjective map of cochain complexes with kernel $I^\bullet$, so that we have a short exact sequence of complexes

$$0 \to I^\bullet \to A^\bullet \to B^\bullet \to 0.$$

In the derived category this realizes $A^\bullet$ as the homotopy kernel of a map $B^\bullet \to I^\bullet[1]$. Therefore if the image of

$$H^0(B^\bullet) \to H^0(I^\bullet[1]) = H^1(I^\bullet)$$

is nonzero, then it gives an obstruction to lifting elements from $H^0(B^\bullet)$ to $H^0(A^\bullet)$.

Now we want to construct a nonabelian version of this. The miracle of derived deformation theory is that **tangent spaces are obstruction spaces**. (This also covers the classical situation, where if the tangent space is a cohomology group, the obstruction space tends to be the next cohomology group up.)

4.2.1 Construction for cdga's

We want an analogue of the homological construction from above. Let $A_\bullet, B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$, with a square-zero extension $f : A_\bullet \twoheadrightarrow B_\bullet$, i.e. a level-wise surjection such that the kernel I squares to zero. Now I is an ideal in $A_\bullet$ and so we have a natural inclusion $I_\bullet \hookrightarrow A_\bullet$. Write

$$\widetilde{B}_\bullet \coloneqq \mathrm{Cone}(I_\bullet \hookrightarrow A_\bullet).$$

In the case where A, B are commutative algebras, then we can realize $\widetilde{B}_\bullet$ concretely as the complex

$$A \hookleftarrow I \leftarrow 0 \cdots .$$

Then $\widetilde{B}_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ with the obvious multiplication. We also clearly have a quasi-isomorphism

$$\widetilde{B}_\bullet \xrightarrow{\sim} B_\bullet.$$

Generally speaking, $A_\bullet$ is some extension of $I_\bullet$ and $B_\bullet$, but not free; so we can compare $A_\bullet$ with the split extension $B_\bullet \oplus I_\bullet[1]$. We have an obvious map

$$v : B_\bullet \to B_\bullet \oplus I_\bullet[1], \quad b \mapsto (b, 0).$$

We have another map

$$u : \widetilde{B}_\bullet \to B_\bullet \oplus I_\bullet[1]$$

just killing $I_\bullet \subset A_\bullet$. Notice that as complexes we have

$$A = \widetilde{B}_\bullet \times_{u,(B_\bullet \oplus I_\bullet[1]),v} B_\bullet,$$

so in $\mathrm{dg}_+\mathrm{Alg}_k$ we have

$$A = \widetilde{B}_\bullet \times^h_{u,(B_\bullet \oplus I_\bullet[1]),v} B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k.$$

Now for sufficiently nice functors - such as "half-exact" functors (see [EP21, Definition 3.23]; this mirrors the classic conditions on deformation functors), we can use this to generate obstructions to lifting elements $F(B)$ to $F(A)$. It is known, for example, that any representable functor F on $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$ is half-exact, giving a large class of important examples.

Let F be a half-exact functor, for example a representable functor on $\mathrm{Ho}(\mathrm{dg}_+\mathrm{Alg}_k)$, which are the functors associated to derived affine schemes. Then the expression

$$A = \widetilde{B}_\bullet \times^h_{u,(B_\bullet \oplus I_\bullet[1]),v} B_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$$

writes A as a homotopy fiber product. One of the conditions of being half-exact is that

$$F(A) \twoheadrightarrow F(\widetilde{B}_\bullet) \times_{u,F(B_\bullet \oplus I_\bullet[1]),v} F(B_\bullet) \simeq F(B_\bullet) \times_{u,F(B_\bullet \oplus I_\bullet[1]),v} F(B_\bullet).$$

Now we have two maps

$$u, v : F(B_\bullet) \to F(B_\bullet \oplus I_\bullet[1]).$$

But map v sends $x \mapsto (x, 0)$. Therefore the map

$$u : F(B_\bullet) \to F(B_\bullet \oplus I_\bullet[1])$$

is such that

$$u(x) = (x, 0) \iff x \in \mathrm{Im}(F(A_\bullet) \to F(B_\bullet)).$$

In other words, by allowing ourselves to work with $\mathrm{dg}_+\mathrm{Alg}_k$ (instead of Alg_k), we are guaranteed an obstruction theory

$$(F(B_\bullet \oplus I_\bullet[1]), u)\,.$$

This is in contast to classical deformation theory, where obstruction spaces are not guaranteed (sometimes they exist only to varying levels of "success").

> **Remark 4.23:** We should think about $F(B_\bullet \oplus I_\bullet[1])$ as a "higher degree tangent space." Namely, after switching to tangent complexes instead of tangent spaces, this will become the first cohomology group H^1.

4.3 Postnikov towers

The main purpose of this subsection is to justify the idea that the geometric part of a cdga is mostly just the scheme $\operatorname{Spec} H_0(-)$, with the "rest of the complex" just being "infinitesimal." The basic idea is to write a cdga $A_\bullet$ as limit of a sequence of homotopy square-zero thickenings.

Definition 4.24: Fix $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$. Then define the family of cdgas $(P_nA)_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ (also known as the **Moore-Postnikov tower**) by

$$(P_nA)_i = \begin{cases} A_i & i \leq n, \\ \operatorname{im}(A_{n+1} \xrightarrow{\delta} A_n) & i = n+1, \\ 0 & i > n+1. \end{cases}$$

These form the diagram

$$\begin{tikzcd} & A_\bullet \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] \arrow[drrrr] & & & & & \\ \cdots \arrow[r] & (P_4A)_\bullet \arrow[r] & (P_3A)_\bullet \arrow[r] & (P_2A)_\bullet \arrow[r] & (P_1A)_\bullet \arrow[r] & (P_0A)_\bullet \arrow[r] & 0. \end{tikzcd}$$

The key idea is that if we define

$$(Q_nA)_i = \begin{cases} A_i & i < n, \\ \operatorname{coker}(A_{n+1} \xrightarrow{\delta} A_n) & i = n, \\ 0 & i > n \end{cases}$$

then we obtain the factorization

$$\begin{tikzcd} (P_nA)_\bullet \arrow[rr] \arrow[dr, two heads, "\sim"'] & & (P_{n-1}A)_\bullet \\ & (Q_nA)_\bullet \arrow[ur, "\text{square-zero}"'] & \end{tikzcd}$$

where $(P_nA)_\bullet \to (Q_nA)_\bullet$ is a trivial fibration (it is clearly surjective in all degrees, and clearly a quasi-isomorphism) and $(Q_nA)_\bullet \to (P_{n-1}A)_\bullet$ is a square-zero extension with kernel $H_n(A_\bullet)[-n]$. This means that **we can view** $\operatorname{Spec} A_\bullet$ **as a formal infinitesimal thickening** of $\operatorname{Spec} H_0(A_\bullet)$, as the rest of it is just a bunch of square-zero extensions and trivial fibrations.

If we assume some finiteness conditions, then this viewpoint strengthens to a true formal completion (in the ring-theoretic sense).

Proposition 4.25: Suppose $A_\bullet \in \mathrm{dg}_+\mathrm{Alg}_k$ is such that A_0 is Noetherian and each A_n is a finitely-generated A_0-module. Let $I \coloneqq \ker(A_0 \to H_0(A_\bullet)) \simeq \operatorname{im}(A_1 \to A_0)$. Then the natural map

$$A_\bullet \to \widehat{A_\bullet} \coloneqq \varprojlim_n A_\bullet / I^n A_\bullet$$

is a quasi-isomorphism.

Proof. This is standard commutative algebra; the only thing we really need to check is that the differential behaves nicely all the way through. This is because I is (as a complex) concentrated in degree 0 and therefore the action of completing is more or less just dealing with each graded piece A_k one at a time, i.e. $(\widehat{A_\bullet})_k = \widehat{A_k} \coloneqq \varprojlim_n A_k / I^n A_k$.

If A_0 is Noetherian, then

$$A_0 \to \widehat{A_0} \coloneqq \varprojlim_n A_0 / I^n A_0$$

is flat. Therefore we can write the completion of each A_k (using the fact that they are finitely-generated A_0-

modules) as

$$\widehat{A_k} \coloneqq \varprojlim_n A_k / I^n A_k \simeq A_k \otimes_{A_0} \widehat{A_0}.$$

It follows that

$$H_k(\widehat{A_\bullet}) \simeq H_k(A_\bullet) \otimes_{A_0} \widehat{A_0} = \varprojlim_n \ker(A_k \to A_{k-1}) / (\operatorname{im}(A_{k+1} \to A_k) + I^n \cdot \ker(A_k \to A_{k-1})) .$$

But in fact the extra I^n component does nothing, because *it's already contained in the image.* Let $i \in I \supset I^n$, with $i = \delta(x)$, $x \in A_1$. Then let $y \in \ker(A_k \to A_{k-1})$. By the Leibniz rule of multiplication we have

$$i \cdot y = \delta(x) \cdot y = \delta(x \cdot y) \pm \underbrace{x \cdot \delta(y)}_{=0} = \delta(x \cdot y) \in \operatorname{im}(A_{k+1} \to A_k).$$

It follows that

$$H_k(\widehat{A_\bullet}) \simeq H_k(A_\bullet) \otimes_{A_0} \widehat{A_0} \simeq \varprojlim_n \ker(A_k \to A_{k-1}) / \operatorname{im}(A_{k+1} \to A_k) = H_k(A_\bullet),$$

hence

$$A_\bullet \to \widehat{A_\bullet}$$

is a quasi-isomorphism. □

References

[EP21] Jon Eugster and Jon P Pridham. An introduction to derived (algebraic) geometry. *arXiv preprint arXiv:2109.14594*, 2021.